Deriving the Hyperbolic Trig Functions

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HYPERBOLIC FUNCTIONS

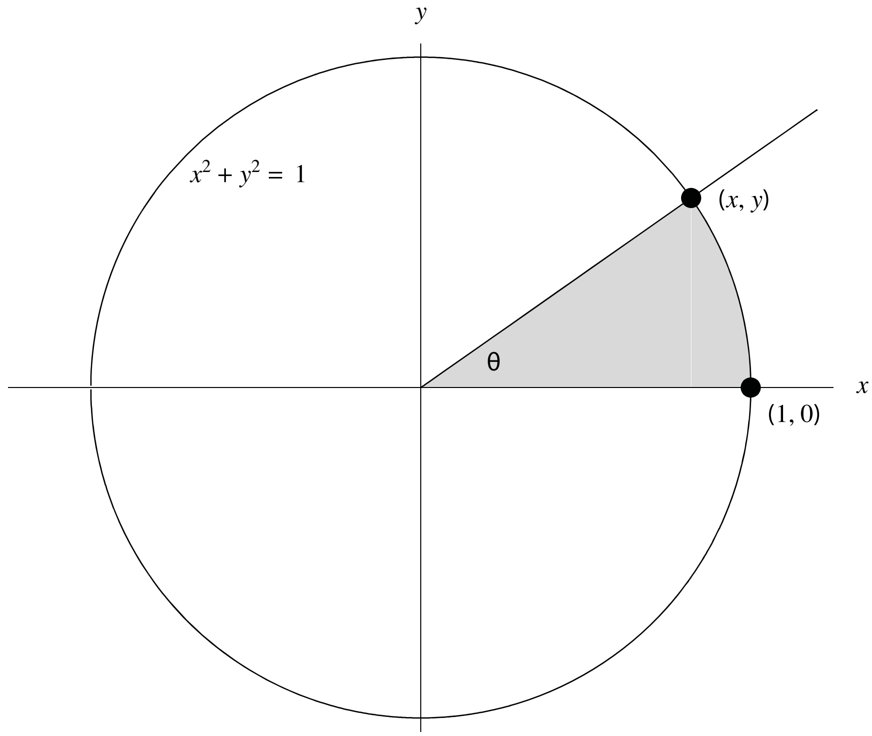
(based on a worksheet by Steve Condie)

Part I. Definitions

If we graph the "unit circle" with center at the origin, the trigonometric functions sine and cosine can be defined in terms of the coordinates on the circle. From the figure below we see that

$$\cos\theta = x \text{ and } \sin\theta = y.$$

We can find $\cos u$ and $\sin u$ for any real number *u* in this way.



Alternately, again referring to the figure above, the area of the shaded portion of the circle is given by

$$\begin{aligned} A &= \frac{\theta}{2\pi}\cdot(\text{area of the circle}) \\ &= \frac{\theta}{2\pi}\cdot\pi r^2 = \frac{\theta}{2}\cdot 1^2 \\ &= \frac{\theta}{2} \end{aligned}$$

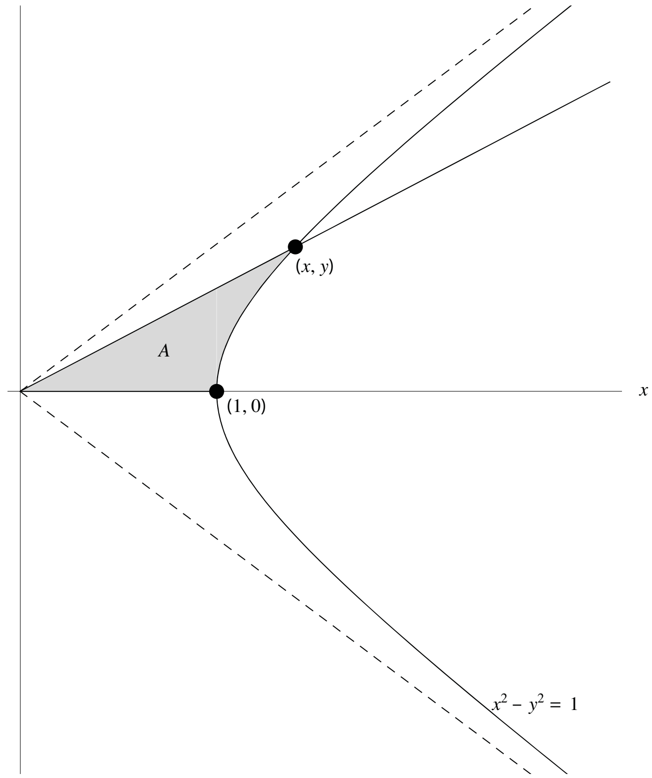
We now let (x, y) be the point of intersection of the unit circle and the terminal side of the angle that sweeps out an area *A*. Let $u = 2A$ and define cosine and sine by:

$$\cos u = x \text{ and } \sin u = y.$$

If the angle sweeping out *A* is measured in the clockwise direction from the positive *x*-axis, we take $u = -2A$. In this manner we again have $\cos u$ and $\sin u$ defined for all real numbers *u*. You should convince yourself that the definitions here are equivalent to our original definitions of sine and cosine.

Suppose we now sketch the graph of a "unit hyperbola"—that is, the hyperbola which has center $(0,0)$, vertex $(1,0)$, and asymptotes with slope ± 1. We look at the right branch of this hyperbola $x^2 - y^2 = 1$.

We define the hyperbolic sine and hyperbolic cosine as follows. (Remark: Sometimes sine and cosine are called circular functions. Note the similarity in the way these functions are defined on the hyperbola.)



Draw a line from the origin to a point (x, y) on the right branch of the hyperbola, (see Figure 2). Let *A* be the area of the region enclosed by the line from the origin to the point (x, y), the *x*-axis, and the hyperbola. If $u = 2A$, we define the hyperbolic sine (sinh) and hyperbolic cosine (cosh) by

$$\cosh u = x \text{ and } \sinh u = y.$$

We take *u* to be negative if the point (x, y) is below the *x*-axis.

Use the graph to answer the following:

a. $\sinh 0 =$ ______ b. $\cosh 0 =$ ______

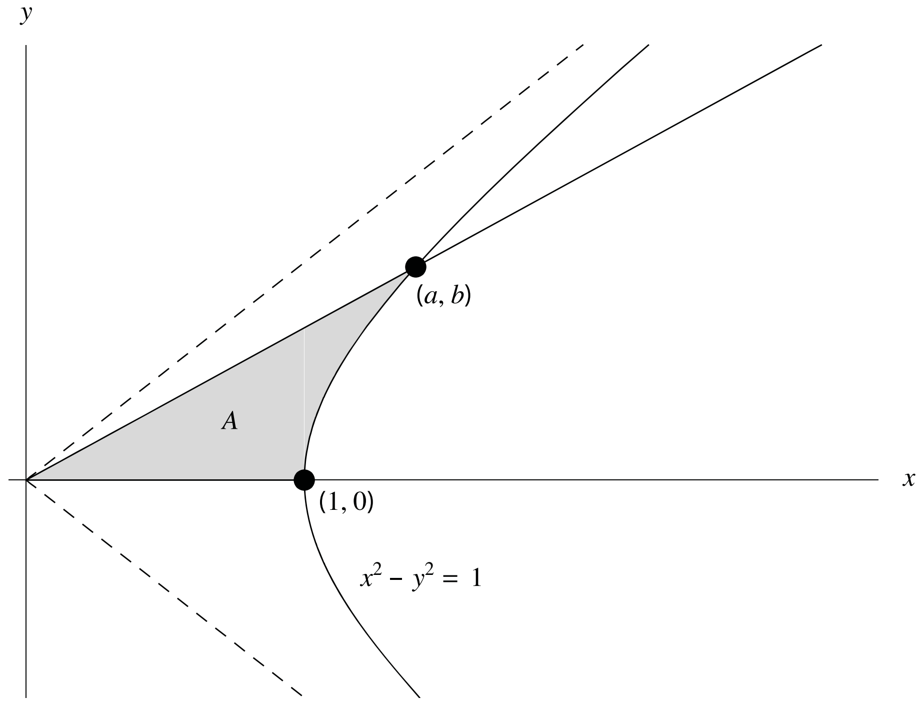
c. Classify sinh and cosh as odd, even or neither. Explain your answers.

d. Find an identity involving $\cosh^2 u$ and $\sinh^2 u$.

Part II. Evaluating Hyperbolic Sine and Cosine

As might be expected, these definitions are quite cumbersome to work with should we wish to find the value of $\sinh 2$ or $\cosh 2$. However, with a little help from a CAS we can find a formula for $\sinh u$ and $\cosh u$ which will be relatively simple to use.

a. Let A be the area of the shaded region pictured in the figure below. Thinking of the right branch of the hyperbola as a function of y, we have that $x =$ ___________ . (Why is this a useful way to approach this problem?)



b. Express the area of A as an integral: $A =$ ____________________ .

c. Use your computer algebra system to calculate this integral:
(If you are looking for a challenge, try this integral by hand!)

$A =$ ____________________________ . Therefore $u = 2A =$ __________________________ .

d. From the definition of sinh and cosh, it follows that

i. $\sinh u = \sinh\left(\rule{4cm}{0.4pt}\right) = b$, and

ii. $\cosh u = \cosh\left(\rule{4cm}{0.4pt}\right) = \sqrt{b^2 + 1}$

Have we accomplished anything? (Although this is a rhetorical question, feel free to give it some thought). Are we any closer to finding $\sinh 2$ or $\cosh 2$?

e. From part c, we have that $u = \ln\left(\sqrt{b^2+1}+b\right)$. Since $\sinh u = b$, solving this equation for *b* will give a formula for $\sinh u$. Solve now:

Therefore,

$$\sinh u = b =$$

f. From part c, we also have that $\cosh u = \sqrt{b^2+1}$. Substitute your value for *b* from above and simplify to get a formula for $\cosh u$.

You should have that

$$\cosh u =$$

At some point during the above derivation there should have been an audible exclamation of astonishment! We first defined the hyperbolic sine and cosine in terms of points on the "unit" hyperbola. What would lead us to believe that these functions would be so intimately intertwined with the exponential function e^u? (If you don't have e^u involved in your formulae above, check with your classmates). Formulate a question involving the circular trig functions, the hyperbolic trig functions, the exponential function, and the derivation above; or connections among these. Write that question here.

Part III. Derivatives of Hyperbolic Sine and Cosine

We would like to find the derivatives of these functions. Use the formulae you derived in Part II to find the derivatives of sinh and cosh in terms of sinh and cosh.

a. Find $D_x(\sinh x)$.

(Hmm? this looks familiar. Apparently the derivative of sinh has a striking similarity to the derivative of sine.)

b. Find $D_x(\cosh x)$.

Part IV. The Other Hyperbolic Trig Functions

Since sinh and cosh are defined in a analogous manner to sine and cosine, we can quite logically define four more hyperbolic functions as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Using the quotient rule along with your results from Part III, find

a. $D_x(\tanh x)$

b. $D_x(\coth x)$

c. $D_x\left(\operatorname{sech} x\right)$

d. $D_x\left(\operatorname{csch} x\right)$

From the above work we immediately have:

e. $\int \sinh x\, dx =$ ____________________

f. $\int \cosh x\, dx =$ ____________________

g. $\int \operatorname{sech}^2 x\, dx =$ ____________________

h. $\int \operatorname{csch}^2 x\, dx =$ ____________________

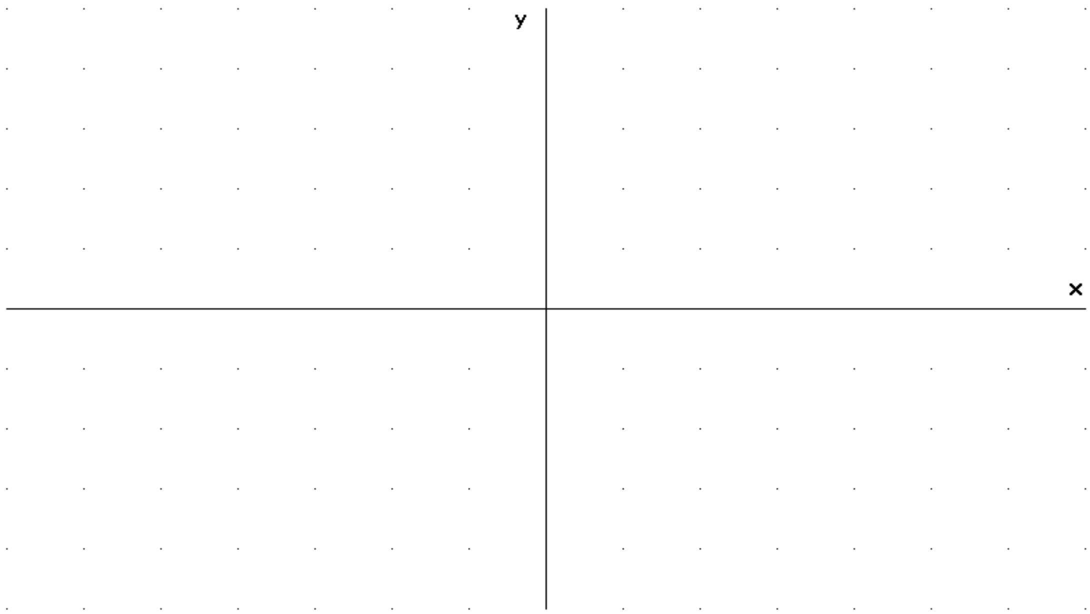
i. $\int \operatorname{sech} x \tanh x\, dx =$ ____________________

j. $\int \operatorname{csch} x \coth x\, dx =$ ____________________

Try to derive some other “hyperbolic trig” identities.

Part V: Graphs

Use your calculator or what you know from graphing with help of the derivative, to sketch the graphs of $y = \cosh x$ and $y = \sinh x$.



Use your calculator to graph $y = \sinh x$, $y = \ln\left(x + \sqrt{x^2 + 1}\right)$, and $y = x$ on the same set of coordinate axes. What does your graph suggest? Would you have expected this? Why or why not?