# Trigonometric Identities: Techniques, Examples, Connections

Isaac Greenspan
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http://ilg.trica.com/talks/

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In case you get bored:
Prove: if $a + b + c = \pi$ and $n$ is an integer, then
$\tan(na) + \tan(nb) + \tan(nc) = \tan(na)\tan(nb)\tan(nc)$

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Need more? How about the other 4 trig functions?


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# Techniques

Prove $1+\tan^2 x = \sec^2 x$ by:

a. Rewriting one side using definitions, known identities, and algebraic properties until it equals the other side.

b. Rewriting each side independently until equal expressions are obtained.

c. Beginning with a known identity and transform it until the desired identity appears.

d. Transforming both sides of the equation to be proved using *reversible steps* until a known identity appears.

[UCSMP Precalculus and Discrete Mathematics, 2nd ed.]

# Techniques

Proof Technique Summary

a. LHS = ⋯ = RHS

b. LHS = ⋯ = ▢; RHS = ⋯ = ▢

c. Known Identity ⇒ Target Identity

d. Target Identity ⇒ Known Identity (with reversible steps)

# Techniques

For each of the following identities, choose a method of proof.

$$\frac{1-\sin t}{\cos t} = \frac{\cos t}{1+\sin t}$$

$$\frac{1}{1+\sin m} + \frac{1}{1-\sin m} = 2\sec^2 m$$

$$\frac{\sin t+\cos t}{\sec t+\csc t} = \frac{\cos t}{\csc t}$$

$$\frac{1}{\sqrt{1+\sin x}} = \left|\sec x\right|\sqrt{1-\sin x}$$

$$\tan t + \cot t = \sec t \csc t$$

$$\sin^4 t - \cos^4 t = \sin^2 t - \cos^2 t$$

$$1-2\cos^2 r = 2\sin^2 r - 1$$

$$\tan^4 p - 1 = \sec^4 p - 2\sec^2 p$$

[Dolciani Modern Introductory Analysis]

# Connections

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So $$R_\theta \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

and $$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

# Connections

Rotating by $\alpha$ then by $\beta$ should be the same as rotating by $(\alpha + \beta)$. Use this fact to derive identities for sine and cosine of $(\alpha + \beta)$.

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Did you use matrices or
did you use complex numbers?

# Connections

Hyperbolic Trigonometry:

- Refer to your handout
- Look at review of circular trigonometry—you need this particular perspective on circular trigonometry for the derivation of hyperbolic trigonometry.
- Go through derivation of hyperbolic trigonometry. You may want to let a CAS do the heavy lifting for you and/or skip over some of the calculus involved.

# Connections

## Hyperbolic Trigonometry—Executive Summary

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

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Use your formulae to prove $\cos 2x = 2\cos^2 x - 1$ (hint: start with the right side).

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Use your formulae to evaluate $\int \cos^2 x \, dx$ without using the normal reduction formula.

Write $\cosh ix$ and $\sinh ix$ in terms of $\cos x$ and $\sin x$.

# Examples

## sin($nx$) and cos($nx$)

# Examples

If $\alpha + \beta + \gamma = \pi$...

2.7182818284590452353602874713526624
9775724709369995957496696762772407 6
6303535475945713821785251664274274 6
6391932003059921817413596629043572 9
0033429526059563073813232862794349 0
7632338298807531952510190115738341 8
7930702154089149934884167509244761 4
6066808226480016847741185374234544 2
4371075390777449920695517027618386 0

Yes, the background is the first several digits of e.

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But, really, you should be wondering why this presentation isn't over yet…

2.718281828459045235360287471352662 4
97757247093699959574966967627724076
63035354759457138217852516642742746
63919320030599218174135966290435729
00334295260595630738132328627943490
76323382988075319525101901157383418
79307021540891499348841675092447614
60668082264800168477411853742345442
43710753907774499206955170276183860

2.7182818284590452353602874713526624
97757247093699959574966967627724076
63035354759457138217852516642742746
63919320030599218174135966290435729
00334295260595630738132328627943490
76323382988075319525101901157383418
79307021540891499348841675092447614
60668082264800168477411853742345442
43710753907774499206955170276183860